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Constant Tangential Low-Thrust Trajectories near an Oblate Planet

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The problem of changing spacecraft energy in minimum time near an oblate planet by means of tangential low thrust is considered. The mass flow rate and the exhaust velocity are assumed constant. Simple approximate formulas allowing calculation of the spacecraft three-dimensional spiral trajectory are obtained. Simple first-approximation formulas and more accurate second-approximation formulas are given. All of the formulas are valid both for ascent and descent of the spacecraft orbit. The accuracy of the formulas is estimated; it is shown that they provide high accuracy when the thrust acceleration is much smaller than the gravitational acceleration.

Nomenclature

a = semimajor axis
c = exhaust velocity
E = energy constant
g = gravitational acceleration at distance r from the planet
h = $|\boldsymbol{h}|$
$\boldsymbol{h}$ = angular momentum, $\{h_x, h_y, h_z\}$
i = inclination of the orbit
J_2 = coefficient of the second zonal harmonic
m = spacecraft current mass
m_p = consumed propellant mass
N = number of the spacecraft sidereal orbits at time t (fractional)
N_Ω = number of the nodal orbits at time t (fractional)
n = mean motion
P = orbital period
R_e = mean equatorial radius of the planet
r = $|\boldsymbol{r}|$
$\boldsymbol{r}$ = position vector, $\{x, y, z\}$
t = flight time with initial value, 0
U = gravitational potential of the oblate planet
u = argument of latitude
v = speed, $|\boldsymbol{v}|$
$\boldsymbol{v}$ = velocity vector
α = magnitude of the thrust acceleration
$\tilde{\alpha}$ = nondimensional thrust acceleration
β = mass flow rate, $\dot{m}_p > 0$
η = $+$ for ascent, $-$ for descent; $\pm v/c$
μ = gravity constant
Φ = full transfer angle for time t
φ = phase angle $(0 \leq \varphi < 2\pi)$, $\mathrm{mod}_{2\pi}\,\Phi$
Ω = longitude of the ascending node

Subscripts

par = values of parameters in the parabolic orbit
Ω = values of parameters in the ascending node
0 = values of parameters at $t = 0$ or zero-order integrals L_k, l_k
* = parameters in initial or final orbit for transfer to or from infinity, respectively

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Superscripts

· = derivative with respect to time
— = parameters averaged over one orbit
ˆ = second-approximation parameter value

Introduction

THE main difficulty in the calculation of a low-thrust transfer near a planet is the large number of revolutions around the planet the spacecraft performs in a spiral trajectory, unlike an interplanetary transfer. This is caused by the low magnitude of the thrust with respect to the planet's gravitational force. The spiral low-thrust transfer near a planet has been studied by a number of authors[1–28] (only a representative sample). The problem in different formulations was solved for various models and approaches in Refs. 1–28, for the power-limited propulsion and the constant exhaust velocity cases, for an optimal thrust direction and the tangential direction, for different transfer types (circular, elliptical, and transfer outside the sphere of influence, etc.). The papers considered an inverse-square gravity field, except for Refs. 22, 23, and 25, where an oblate planet was considered. Most of the solutions are rather complicated due to the difficulty of the problem.

However, in some cases simplifying assumptions can be made, and a simple approximate solution convenient for a quick qualitative analysis can be obtained. Initially, this was done in Refs. 1, 3, and 10 for the orbital energy change by tangential thrust and for a circular initial orbit. Very simple expressions for the orbital parameters derived in these papers provide an accuracy sufficient for the early phase of low-thrust mission analysis. However, simple solutions were obtained in these references and other publications for an inverse-square field only. At the same time, secular perturbations of the transfer trajectory caused by the planet's oblateness must be taken into account even at the early phase of the mission design. The oblateness was taken into consideration in Refs. 22, 23, and 25, but only for power-limited propulsion and only for a transfer between neighboring orbits.

The present paper offers simple formulas for approximate calculation of the spacecraft spiral trajectories around an oblate planet for the orbital energy change. The following assumptions are made:

1) No other external forces except the gravity field of the oblate planet act on the spacecraft.

2) The thrust is tangential and constant; this assumption is not quite correct in the case where solar electric power is used near a planet with a significantly eccentric orbit (such as Mercury or Mars). However, we neglect variations of the electric power due to the eccentricity.

3) Exhaust velocity is constant.

4) Thrust without coast arcs is considered, that is, minimum-time transfer is assumed.

The second and third assumptions give $\beta =$ const and $c =$ const, and hence,

$$\alpha = -\dot{m}c/m = \beta c/(m_0 - \beta t), \qquad \Delta v = c\ln[m_0/(m_0 - \beta t)] \tag{1}$$

Two sets of the formulas are obtained under these assumptions: a simple first approximation, which is similar to the solutions given in Refs. 1, 3, and 10, and a more complicated second approximation, which more accurately takes into account the oblateness.

First Approximation

Analysis

The first approximation for all parameters except the longitude of the ascending node will be obtained neglecting the oblateness, that is, for the inverse-square law. Using the expression for energy $v^2 = 2\mu/r - \mu/a$, we obtain the change of the semimajor axis caused by the tangential thrust acceleration

$$\pm 2v\alpha = (\mu/a^2)\dot{a} \tag{2}$$

where + corresponds to ascent and − to descent. An equation for the eccentricity must be added to Eq. (2) to form a closed equation system. The solution of this system gives the spiral transfer including the periodical perturbations, but the solution is rather complicated. However, it can be significantly simplified by taking into account the following. Because the thrust is low, the osculating orbit is nearly circular. Therefore, let us assume that it remains circular during the transfer, that is,

$$r = a \tag{3}$$

$$v = \sqrt{\mu/r} \tag{4}$$

Under these assumptions, Eq. (2) is closed. Let us define the nondimensional variable

$$\eta = \pm(v/c) \tag{5}$$

where + corresponds to ascent and − to descent. Integration of Eq. (2) using Eqs. (1) and (3–5) gives

$$r = r_0\{1 + (1/\eta_0)\ln[1 - (\beta t/m_0)]\}^{-2} \tag{6}$$

This equation for ascent was obtained first by Tsien,[1] then Melbourne[3] derived it in a way similar to the one used here. Transfer time between two circular orbits of given radii r_0 and r can be easily found from Eq. (6):

$$t = (m_0/\beta)[1 - \exp(\eta - \eta_0)] \tag{7}$$

Comparison of Eqs. (1) and (7) gives an interesting expression for the characteristic velocity:

$$\Delta v = |v - v_0| \tag{8}$$

Note that because $v_0 > v$ for ascent and $v_0 < v$ for descent and due to Eq. (8)

$$\eta - \eta_0 = -|v - v_0|/c = -(\Delta v/c) \tag{9}$$

Now consider transfer and phase angles and the number of sidereal orbits. The difference between the nodal and sidereal periods is caused by the permanent shift (precession) of the ascending node due to the oblateness and is rather significant. For instance, it is about 3 s for a satellite in low Earth orbit with an inclination of 51.6 deg. The transfer angle for the circular osculating orbit is

$$\Phi = 2\pi N = \int_0^t n\,\mathrm{d}t \tag{10}$$

where

$$n = \sqrt{\mu}/r^{\frac{3}{2}} \tag{11}$$

Then from Eqs. (10) and (11) we have

$$\Phi = \sqrt{\mu}\int_0^t \frac{\mathrm{d}t}{r^{\frac{3}{2}}} \tag{12}$$

The integral in Eq. (12) is calculated in Appendix A [see Eq. (A9)]. Finally, the full transfer angle is

$$\Phi = (c^3/\mu\beta)|L_3| \tag{13}$$

where the integral L_k is defined and calculated in Appendix A. Because of Eq. (10), the number of sidereal orbits is

$$N = (c^3/2\pi\mu\beta)|L_3| \tag{14}$$

The phase angle is

$$\varphi = \mathrm{mod}_{2\pi}\,\Phi = 2\pi(N - \mathrm{int}\,N) \tag{15}$$

For the first time an expression for the full transfer angle was obtained in Ref. 10, although only for ascent and in a form less convenient for calculations than in the present paper.

The precession rate of the ascending node for the circular orbit can be written as

$$\dot{\Omega} = -\tfrac{3}{2}J_2 n(R_e/r)^2\cos i \tag{16}$$

For the calculation of the number of orbits, we can obtain from Eq. (16)

$$\Omega = \Omega_0 - \tfrac{3}{2} J_2 \left(R_e^2 c^7 / \mu^3 \beta \right) \cos i \, |L_7| \tag{17}$$

To find the number of the nodal orbits, note that the change of the osculating node-to-node transfer angle for infinitesimal time interval dt is $d\Omega \cos i = \dot{\Omega} \cos i \, dt$. Thus,

$$N_\Omega = N + (\Omega - \Omega_0) \cos i / 2\pi \tag{18}$$

The argument of latitude can be derived from Eqs. (15) and (18):

$$u = u_0 + \varphi - (\Omega - \Omega_0) \cos i \tag{19}$$

Now the components of the spacecraft position and velocity in the Cartesian coordinate system can be calculated as follows:

$$x = r(\cos\Omega \cos u - \sin\Omega \sin u \cos i)$$

$$y = r(\sin\Omega \cos u + \cos\Omega \sin u \cos i), \qquad z = r \sin u \sin i$$

$$v_x = v(-\cos\Omega \sin u - \sin\Omega \cos u \cos i)$$

$$v_y = v(-\sin\Omega \sin u + \cos\Omega \cos u \cos i), \qquad v_z = v \cos u \sin i \tag{20}$$

Accuracy of the Formulas

Consider a spacecraft ascent near Earth with initial values $r_0 = 6771$ km, $c = 15$ km/s, $\alpha_0 = 5 \times 10^{-5}$ g_e where g_e = 9.8066 m/s^2 is the Earth gravitational acceleration at sea level; the final radius is selected as 10^6 km. The fourth-order Runge–Kutta propagator with variable step equivalent to 1-deg arc of the spiral was used for the accurate calculations here and to follow.

Let us consider the inverse-square-law central force field neglecting the oblateness effects. Figures 1 and 2 show maximum absolute errors in r and φ over one revolution vs time [the errors are the differences between the parameters calculated using Eqs. (6) and (15) and those obtained from numerical integration]. The errors include mean error and maximum amplitude of the periodic perturbations. The accurate value of r also is given in Fig. 1. As is seen in Fig. 1, the absolute value of the maximum radius calculation error is within 1 km up to 40 days in the transfer and within 5 km near geosynchronous orbit ($r \approx 42{,}000$ km). Figure 2 also shows the accuracy of the phase angle (within 1 deg) for 100 days of flight, that is, up to 60,000 km. The number of orbits is calculated accurately for all transfer distances (maximum error is reached for $r = 10^6$ km and equal to 0.2 of a revolution). These examples demonstrate that the simple first-approximations are applicable for the inverse-square gravity field and that the periodical perturbations can be ignored for approximate calculations.

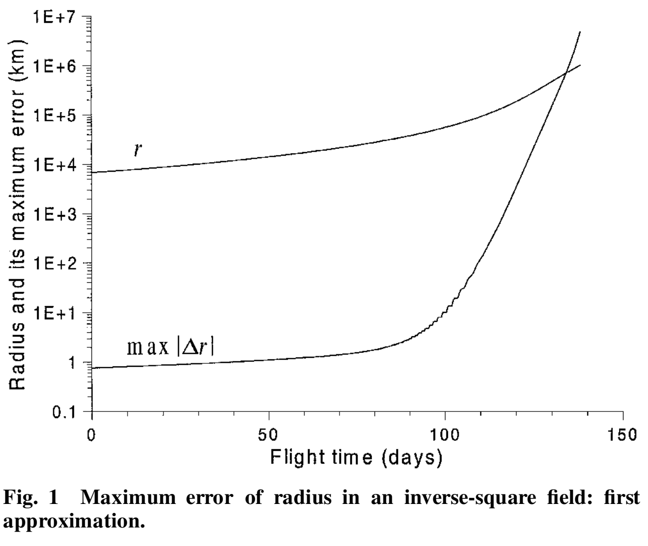

Fig. 1 Maximum error of radius in an inverse-square field: first approximation.

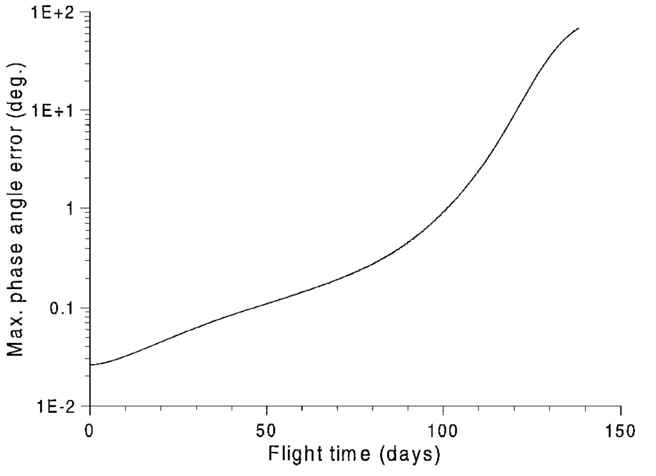

Fig. 2 Maximum error of phase angle in an inverse-square field: first approximation.

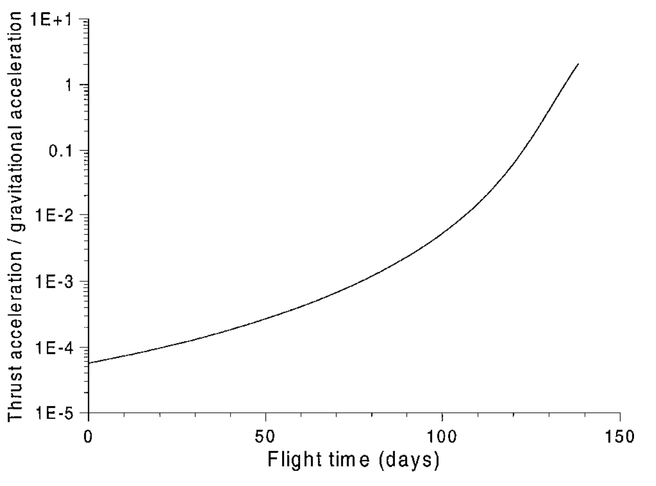

Fig. 3 Nondimensional thrust acceleration.

However, as Figs. 1 and 2 show, after some time the accuracy of the orbit radius and phase angle calculation becomes unsatisfactory. Let us introduce the nondimensional acceleration

$$\tilde{\alpha} = \alpha / g = \alpha r^2 / \mu \tag{21}$$

Figure 3 shows $\tilde{\alpha}$ vs time for the considered example. Comparing Figs. 1 and 2 with Fig. 3, we can see that the calculation accuracy is high for $\tilde{\alpha} \ll 1$.

Now let us take into account the Earth oblateness. It is more convenient to consider the argument of latitude instead of the phase angle in this case. Figures 4–6 represent the maximal (within one revolution) absolute values of the calculation errors for the distance, the argument of latitude, and the longitude of the ascending node for four values of the orbit inclination. The sharp local minima of the curves in Figs. 4 and 5 correspond to the mean error crossing zero. As is seen in Figs. 4 and 5, the errors of the distance and argument of latitude calculations are much bigger in the case of an oblate planet than in the case of an inverse-square gravity field. The first-approximation formulas are not acceptable for calculation of the spacecraft positions near an oblate planet due to the large error in the argument of latitude. Nevertheless, they can be used for some approximate preliminary analysis of the spacecraft motion.

Second Approximation

Analysis

Here the oblateness will be taken into account for all orbital parameters. To derive the formulas of the second approximation, consider the energy relationship for the motion near an oblate planet:

$$v^2/2 = U + E \tag{22}$$

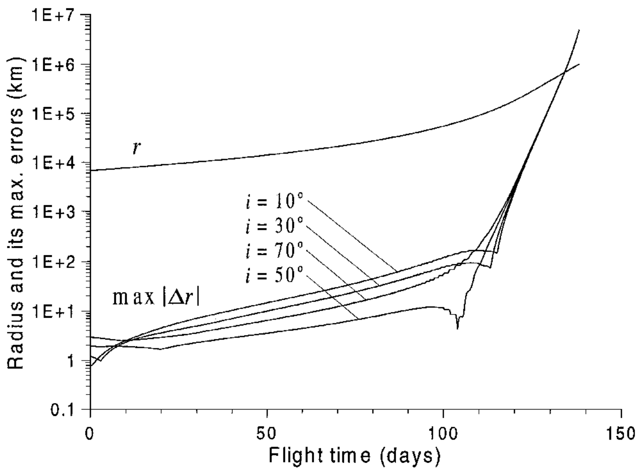

Fig. 4 Maximum errors of radius near an oblate planet: first approximation.

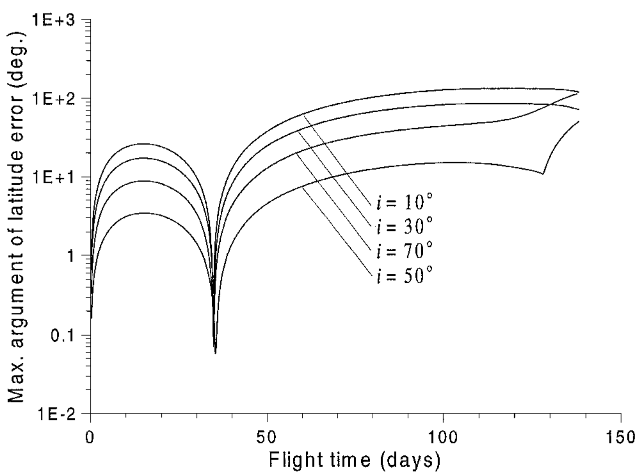

Fig. 5 Maximum errors of argument of latitude near an oblate planet: first approximation.

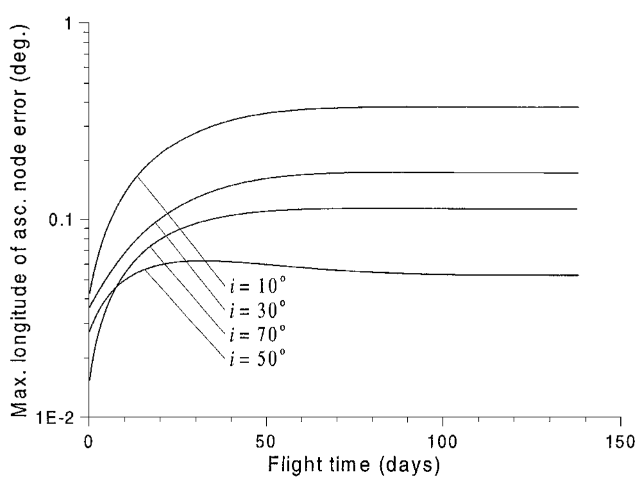

Fig. 6 Maximum errors of longitude of ascending node near an oblate planet: first approximation.

where

$$U = (\mu/r)\left[1 + (J_2/2)(R_e/r)^2(1 - 3\sin^2 i \sin^2 u)\right] \tag{23}$$

is the gravitational potential. Change in the energy constant E caused by the tangential thrust is

$$\pm v\alpha = \dot{E} \tag{24}$$

where + and − correspond to ascent and descent, respectively. Now let us consider the circular orbit, that is, assume $r = \text{const}$ (of course an exact circular orbit inclined to the equator is impossible near an oblate planet, but we neglect periodic perturbations). Let us define the parameter

$$B = J_2 R_e^2\left(1 - \tfrac{3}{2}\sin^2 i\right) \tag{25}$$

The circularity assumption allows the averaging to be performed in terms of either the time or the angle. The potential (23) averaged over one revolution is

$$\bar{U} = (\mu/r)\left[1 + \tfrac{1}{2}(B/r^2)\right] \tag{26}$$

Average gravitational acceleration is given by

$$\bar{g} = \left|\frac{\mathrm{d}\bar{U}}{\mathrm{d}r}\right| = \frac{\mu}{r^2}\left(1 + \frac{3}{2}\frac{B}{r^2}\right)$$

On the other hand, the centripetal acceleration is $\bar{g} = \bar{v}^2/r$; thus, ignoring terms of order of $\mathcal{O}(J_2)$, we have for the circular orbit

$$\bar{v} = \sqrt{\mu/r}\left[1 + \tfrac{3}{4}(B/r^2)\right] \tag{27}$$

Now we can find E from Eqs. (22), (26), and (27):

$$E = -(\mu/2r)(1 - B/2r^2) \tag{28}$$

Substituting Eqs. (27) and (28) into Eq. (24), using Eq. (1), and ignoring $\mathcal{O}(J_2)$, we have

$$\pm(\dot{m}c/m) = -\left(\sqrt{\mu}/2r^{\frac{3}{2}}\right)\left[1 - \tfrac{9}{4}(B/r^2)\right]\dot{r} \tag{29}$$

where + and − correspond to ascent and descent, respectively. Integrating Eq. (29), we obtain

$$\pm c\ln(m/m_0) = \sqrt{\mu}\left(1/\sqrt{r} - 1/\sqrt{r_0}\right) - \tfrac{9}{20}B\sqrt{\mu}\left(1/r^{\frac{5}{2}} - 1/r_0^{\frac{5}{2}}\right) \tag{30}$$

Let us represent the second approximation for the radius in Eq. (30) as

$$\hat{r} = r + \Delta r \tag{31}$$

where r is given by Eq. (6), and resolving Eq. (30) with respect to Δr and ignoring $\mathcal{O}(\Delta r^2)$, after simple transformations we obtain

$$\hat{r} = r\left\{1 + \tfrac{9}{10}(B/r^2)\left[(r/r_0)^{\frac{5}{2}} - 1\right]\right\} \tag{32}$$

The second approximation for the mean motion can be found from Eqs. (27) and (32):

$$\hat{n} = \frac{\bar{v}}{\hat{r}} = \sqrt{\frac{\mu}{\hat{r}^3}}\left(1 + \frac{3}{4}\frac{B}{r^2}\right) = \sqrt{\frac{\mu}{r^3}}\left\{1 + \frac{3}{4}\frac{B}{r^2} - \frac{27}{20}\frac{B}{r^2}\right.$$
$$\left.\times\left[\left(\frac{r}{r_0}\right)^{\frac{5}{2}} - 1\right]\right\} = \sqrt{\frac{\mu}{r^3}}\left\{1 - \frac{3}{20}\frac{B}{r^2}\left[9\left(\frac{r}{r_0}\right)^{\frac{5}{2}} - 14\right]\right\} \tag{33}$$

Now we can find the full transfer angle,

$$\hat{\Phi} = \sqrt{\mu}\int_0^t \hat{n}\,\mathrm{d}t \tag{34}$$

Using Eqs. (A9) and (4), we obtain from Eqs. (33) and (34)

$$\hat{\Phi} = (c^2/\mu\beta)\left[c|L_3| - \tfrac{3}{20}(B/\mu^2)\left(9v_0^5|L_2| - 14c^5|L_7|\right)\right] \tag{35}$$

When Eqs. (14) and (15) are used, the second approximations for the sidereal number of orbits $\hat{N}$ and phase angle $\hat{\varphi}$ can be found.

The first-order value of $\dot{\Omega}$ for the motion without any thrust [assuming that Eq. (16) gives the zero-order value] was obtained, for example, by Brouwer.[28] However, the value for a circular orbit is expressed in Ref. 28 via an osculating semimajor axis, which differs from the orbit radius r. The value of $\dot\Omega$ as a function of r is obtained here in Appendix B, see Eqs. (B3) and (B6). We will consider Eq. (B6) given for the orbital parameters values at the ascending node. Substituting Eq. (32) into Eq. (B3) or (B6) and ignoring terms of $\mathcal{O}(J_2^3)$, we obtain for the low-thrust transfer

$$\hat{\dot\Omega} = -\frac{3}{2}J_2\frac{\sqrt{\mu}R_e^2}{\hat r^{\frac{7}{2}}}\left[1-\frac{3}{2}J_2\left(\frac{R_e}{r}\right)^2(1-b\sin^2 i)\right]\cos i$$

$$= -\frac{3}{2}J_2\frac{\sqrt{\mu}R_e^2}{r^{\frac{7}{2}}}\left\{1-\frac{3}{2}J_2\left(\frac{R_e}{r}\right)^2(1-b\sin^2 i)-\frac{63}{20}\frac{B}{r^2}\right.$$

$$\left.\times\left[\left(\frac{r}{r_0}\right)^{\frac{5}{2}}-1\right]\right\}\cos i = -\frac{3}{2}J_2\frac{\sqrt{\mu}R_e^2}{r^{\frac{7}{2}}}\left\{1-\frac{3}{20}J_2\left(\frac{R_e}{r}\right)^2\right.$$

$$\left.\times\left[21\left(1-\frac{3}{2}\sin^2 i\right)\left(\frac{r}{r_0}\right)^{\frac{5}{2}}-(11-b\sin^2 i)\right]\right\}\cos i \qquad (36)$$

where r is the first-approximation (6), $b=\frac{25}{12}$ if $i=\bar i$ and $b=\frac{31}{12}$ if $i=i_\Omega$, where $\bar i$ and i_Ω are the mean and nodal values of the inclination. Integrating Eq. (36) using Eq. (A9), we have

$$\hat\Omega = \Omega_0 - \frac{3}{2}\frac{J_2R_e^2c^6\cos i}{\mu^3\beta}\left\{c|L_7| - \frac{3}{20}\frac{B}{\mu^2}\left[21v_0^5|L_6| - sc^5|L_{11}|\right]\right\} \qquad (37)$$

where

$$s = \begin{cases} 7+\dfrac{8}{2-3\sin^2 i} & \text{if} \quad i=\bar i \\ \dfrac{2}{9}\left(17+\dfrac{65}{2-3\sin^2 i}\right) & \text{if} \quad i=i_\Omega \end{cases}$$

The longitude of the ascending node also can be expressed via the average values of the orbital parameters using Eq. (B3).

Now the number of nodal orbits $\hat N_\Omega$ and argument of latitude $\hat u$ can be found from Eqs. (18) and (19) in which the second-approximation values of N, Ω, and φ are substituted.

Accuracy of the Formulas

The values of the propulsion and transfer parameters used here are the same as in the preceding section. Figures 7–9 give the maximal (within one revolution) absolute values of the calculation errors in r, u, and Ω given by Eqs. (32), (19), and (37). The sharp local minimum of a curve in Fig. 8 corresponds to the bias crossing zero. Comparison of Figs. 7–9 with Figs. 4–6 shows that the second approximation significantly lowers the calculation errors relative to the first approximation. Figure 10 represents the maximum calculation errors of the spacecraft position given by Eq. (20) together with the orbit radius. As is seen in Fig. 10, the error is within 1% of the radius up to the geosyncronous orbit.

Note that Figs. 7–10 (as well as Figs. 1, 2, and 4–6) give maximum errors that are the sum of the mean error and the maximum amplitude of the periodical components. Thus, the mean errors are smaller.

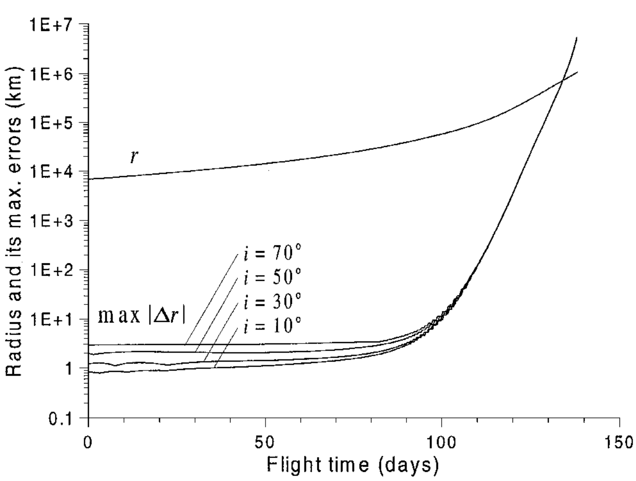

Fig. 7 Maximum errors of radius near an oblate planet: second approximation.

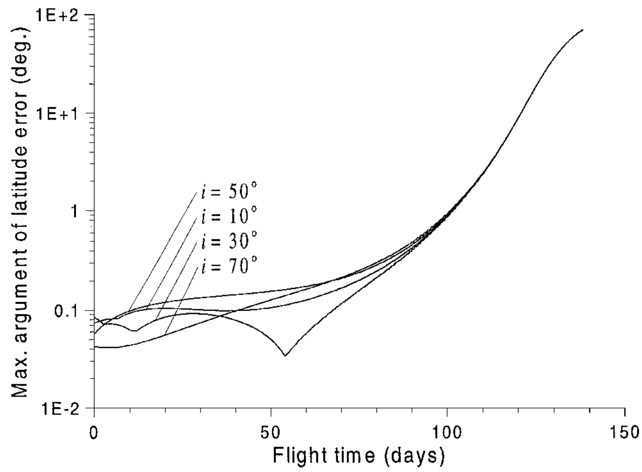

Fig. 8 Maximum errors of argument of latitude near an oblate planet: second approximation.

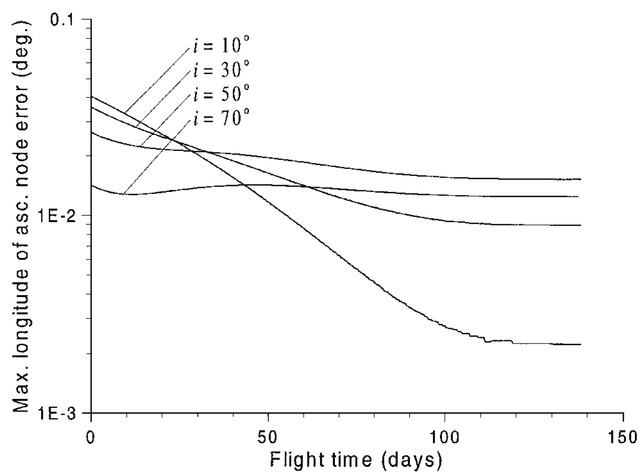

Fig. 9 Maximum errors of longitude of ascending node near an oblate planet: second approximation.

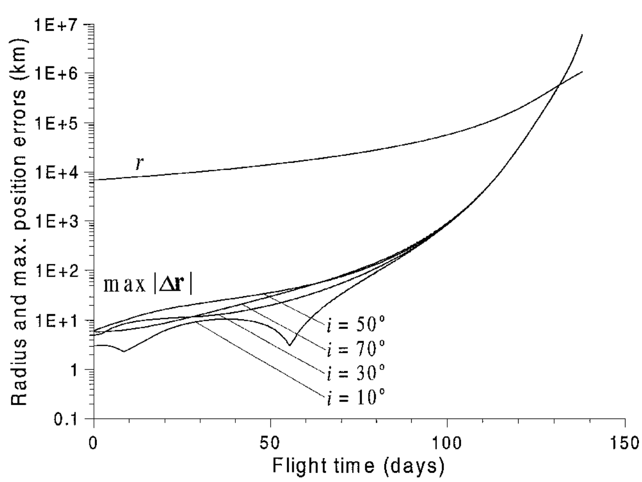

Fig. 10 Maximum errors in position near an oblate planet: second approximation.

Thus, the second approximation gives a much higher accuracy of the spacecraft position than the first approximation. However, note the following: Equation (A8) contains the difference between two close numbers, and accuracy of the calculation of the integral L_k of high order using this equation becomes inadequate for high exhaust velocity. For example, the accuracy of L_{11} is unsatisfactory if $c > 25$ km/s. In this case, it is better to use the simpler formula (17) instead of Eq. (37). As is seen in Fig. 6, formula (17) provides a good accuracy for the longitude of the ascending node calculation (for $i \geq 10$ deg, the error does not exceed 0.4 deg). Another possible way to avoid the accuracy loss is by expanding m in powers of $\eta - \eta_0$ in the calculation of L_k.

Also note that the term J_2^2 in the second approximations is of the same order as J_4. The term J_4 also can be easily taken into account using the equations for secular perturbations.[28]

Transfer Outside the Sphere of Influence

Values of the parameters at infinity and the time to reach parabolic velocity will be determined. Because of the approximate nature of the parameters at infinity and the use of an approximate way to determine the time to reach parabolic velocity, there is no need for the accurate second approximations, and only first-approximation formulas will be used.

Values of the Parameters at Infinity

Let us consider the spacecraft transfer to infinity from a circular orbit of radius r_0 or its insertion into an orbit of radius r from infinity and designate $r_* = r_0$ for ascent to infinity and $r_* = r$ for descent from infinity. The subscript $*$ will designate parameters in the orbit of radius r_*, that is, initial parameters for ascent and final ones for descent. Because at infinity the circular velocity is equal to zero, from Eq. (6) we obtain the time of the transfer to or from infinity:

$$t_\infty = (m_0/\beta)[1 - \exp(-v_*/c)] \tag{38}$$

For ascent, $m_* = m_0$, and for descent, $m_* = m_0 - \beta t_\infty = m_0 \exp(-v_*/c)$; thus, using Eq. (5) we can write Eq. (38) in another form:

$$t_\infty = (m_*/\beta)|1 - \exp(-\eta_*)| \tag{39}$$

Let $P_* = 2\pi\sqrt{(r_*^3/\mu)}$ be the period of the initial or final circular orbit for ascent and descent, respectively. Assume that $v_* \ll c$; then $\exp(-v_*/c) \approx 1 - v_*/c$ and Eq. (39) takes the form

$$t_\infty \approx P_*/2\pi\tilde{\alpha}_* \tag{40}$$

where $\tilde{\alpha}$ is given by Eq. (21). The characteristic velocity (8) is

$$\Delta v_\infty = v_* \tag{41}$$

At infinity, $\eta = 0$, and Eqs. (13) and (A6–A8) give

$$\Phi_\infty = \left(m_* c^3 \big/ \mu\beta\right)\left|6(m_\infty/m_*) + \eta_*^3 - 3\eta_*^2 + 6\eta_* - 6\right| \tag{42}$$

where m_∞ is the spacecraft mass at infinity, that is, the final mass for ascent and the initial one for descent. Note that usually $c > 10$ km/s and, thus, for all inner planets and the moon, $|\eta_*| < 1$ for any values of r_*. Using Eq. (39), we have

$$m_\infty/m_* = (m_* \pm \beta t_\infty)/m_* = \exp(-\eta_*) \approx 1 - \eta_* + \left(\eta_*^2/2\right) - \left(\eta_*^3/6\right) + \left(\eta_*^4/24\right) \tag{43}$$

where $+$ is for descent and $-$ for ascent. Substituting Eq. (43) in Eq. (42) and using Eqs. (1), (5), and (21), we obtain

$$\Phi_\infty = 1/4\tilde{\alpha}_* \tag{44}$$

Because of Eq. (10), the number of the sidereal orbits is

$$N_\infty = 1/8\pi\tilde{\alpha}_* \tag{45}$$

Expression (45) coincides with the empirical formula $N_\infty = 0.04/\tilde{\alpha}_0$ given by Beletsky and Egorov[6] for ascent.

Similarly, using Eqs. (17) and (A6–A8), we have:

$$\Omega_\infty = \Omega_0 - \frac{3}{2}J_2\frac{R_e^2 c^7 m_*}{\mu^3\beta}\cos i\left(5040\frac{m_\infty}{m_*} + \eta_*^7 - 7\eta_*^6 + 42\eta_*^5 - 210\eta_*^4 + 840\eta_*^3 - 2520\eta_*^2 + 5040\eta_* - 5040\right) \tag{46}$$

$$\frac{m_\infty}{m_*} = \exp(-\eta_*) \approx 1 - \eta_* + \frac{\eta_*^2}{2} - \frac{\eta_*^3}{6} + \frac{\eta_*^4}{24} - \frac{\eta_*^5}{120} + \frac{\eta_*^6}{720} - \frac{\eta_*^7}{5040} + \frac{\eta_*^8}{40320} \tag{47}$$

and from Eqs. (46) and (47) using Eqs. (1), (5), and (21)

$$\Omega_\infty = \Omega_0 - \frac{3J_2\cos i}{16\tilde{\alpha}_*}\left(\frac{R_e}{r_*}\right)^2 \tag{48}$$

Time to Reach Parabolic Velocity or to Descend from an Incoming Parabolic Trajectory

The time to reach parabolic velocity is quite important for the transfer design because it gives the propellant consumption necessary for escaping the planet's sphere of influence with zero excess velocity more accurately than the preceding formulas for escape to infinity. This time was analyzed in Refs. 2, 4, 6, and 18. The approach used in this paper does not permit obtaining the time directly because the osculating orbit is assumed circular during the entire transfer. However, some assessments made earlier allow deriving an empirical formula for calculation of the time to reach parabolic velocity (or time to descend from an incoming parabolic trajectory to a circular orbit). Namely, we can expect that the parabolic velocity is reached when $\tilde{\alpha} \sim 1$. Accurate calculations, that is, using the propagator described earlier, show that for $10^{-5} \leq \alpha_0 \leq 10^{-2}$ and $10\,\text{km/s} \leq c \leq 50\,\text{km/s}$ the value of $\tilde{\alpha}$ at which the parabolic velocity is reached lies within the range $0.77 \leq \tilde{\alpha}_{\text{par}} \leq 0.8$. However, taking into account that $\tilde{\alpha}$ changes quickly when $\tilde{\alpha} \sim 1$ (Fig. 3) and the approximate nature of the calculations, we will take for simplicity

$$\tilde{\alpha}_{\text{par}} = 1 \tag{49}$$

Then due to Eq. (21)

$$\alpha_{\text{par}} = \mu\big/r_{\text{par}}^2 \tag{50}$$

On the other hand, we can find

$$\alpha_{\text{par}} = (\beta c/m_{\text{par}}) = (\beta c/m_*)\exp[-(v_* - v_{\text{par}})/c] \tag{51}$$

[Note that in Eq. (51) and subsequently v_{par} is not a parabolic velocity but the circular one at t_{par}]. Equation (50) gives $r_{\text{par}} = \sqrt{(\mu/\alpha_{\text{par}})}$, and using Eq. (51) we have

$$v_{\text{par}} = \sqrt{\frac{\mu}{r_{\text{par}}}} = \sqrt[4]{\frac{\mu\beta c}{m_*}}\exp\left(-\frac{v_* - v_{\text{par}}}{4c}\right) \tag{52}$$

Assume that $v_* - v_{\text{par}} \ll 4c$ and

$$\exp\left(-\frac{v_* - v_{\text{par}}}{4c}\right) \approx 1 \tag{53}$$

Also note that

$$\mu\beta c/m_* = (\mu/r_*)^2(\beta c/m_*)\left(r_*^2\big/\mu\right) = v_*^4\tilde{\alpha}_* \tag{54}$$

then from Eq. (52), using Eqs. (53) and (54), we have

$$v_{\text{par}} \approx v_*\sqrt[4]{\tilde{\alpha}_*} \tag{55}$$

Substituting Eq. (55) in Eq. (7), we obtain an expression for the time to reach parabolic velocity or to descend from an incoming parabola to a circular orbit:

$$t_{\text{par}} \approx m_*/\beta\left\{1 - \exp\left[-(v_*/c)\left(1 - \sqrt[4]{\tilde{\alpha}_*}\right)\right]\right\} \tag{56}$$

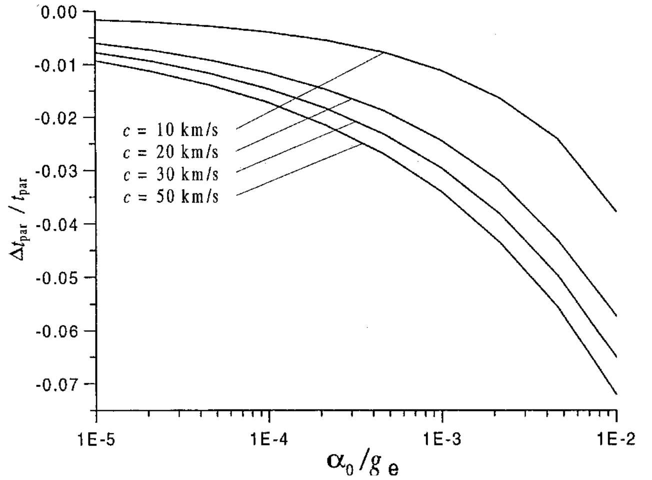

Fig. 11 Errors in time to reach parabolic velocity.

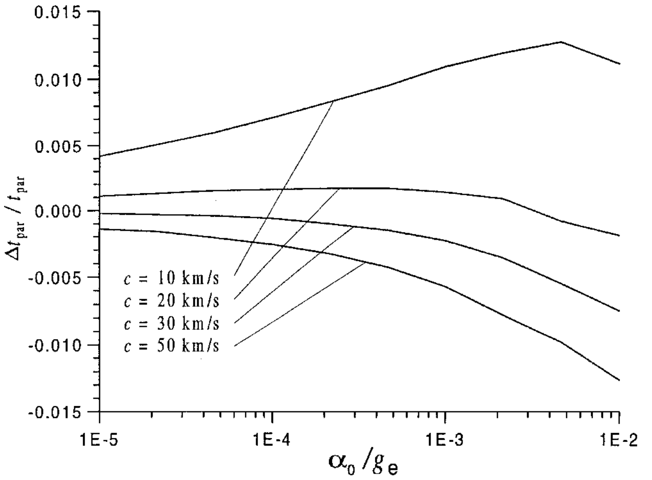

Fig. 12 Errors in time to reach parabolic velocity using correcting factor 0.86.

Figure 11 shows errors of calculations by Eq. (56), that is, differences between the approximate and accurate values, for $10^{-5} \leq \alpha_* \leq 10^{-2} g_e$ and $10 \leq c \leq 50$ km/s. As is seen in Fig. 11, the errors do not exceed 7%.

An empirical correction

$$t_{\text{par}} \approx (m_*/\beta)\left\{1 - \exp\left[-(v_*/c)\left(1 - 0.86\sqrt[4]{\tilde{\alpha}_*}\right)\right]\right\} \tag{57}$$

reduces the errors down to 1.3% as is shown in Fig. 12 (the factor 0.86 was selected to minimize the errors for the thrust and exhaust velocity ranges used). Formulas (56) and especially (57) are more accurate than the corresponding ones given in Refs. 2, 4, 6, and 18, which produce errors up to 10–40%.

From Eqs. (6) and (8), using Eq. (57), we can find the distance to reach parabolic velocity and necessary characteristic velocity:

$$r_{\text{par}} = 1.35\left(r_* \big/ \sqrt{\tilde{\alpha}_*}\right) \tag{58}$$

$$\Delta v_{\text{par}} = v_*\left(1 - 0.86\sqrt[4]{\tilde{\alpha}_*}\right) \tag{59}$$

If $v^* \ll c$, then Eq. (57) can be simplified,

$$t_{\text{par}} \approx P_*/2\pi\tilde{\alpha}_*\left(1 - 0.86\sqrt[4]{\tilde{\alpha}_*}\right) \tag{60}$$

Comparison of Eq. (40) with Eq. (60) shows that, for example, for $\tilde{\alpha}_* \approx 10^{-4}$, the time to reach parabolic velocity differs from the time of escape to infinity by about 10%.

Additional Remarks

Note that because the osculating orbit remains very close to a circular one the proposed formulas for calculation of low-thrust transfers are valid also for circumferencial thrust. However, they can be used also in cases when the low thrust is not tangential but on average is identical to the tangential case. For instance, in Ref. 29, an ascent from low circular orbit of a spin-stabilized spacecraft was considered. There are two 120-deg thrust arcs and two 60-deg coast arcs in every orbit. The thrust direction is constant in each of the thrust arcs and coincides with the velocity direction in the middle of the arc. The average ratio of the tangential thrust component to the thrust value is

$$\frac{3}{2\pi}\int_{-\pi/3}^{\pi/3} \cos\varphi \, \mathrm{d}\varphi = 0.827$$

The radial components on average eliminate each other. When the coast arcs are taken in account, the thrust is approximately equivalent to constant tangential thrust with mass flow rate

$$\bar{\beta} = 0.55\beta$$

where β is real mass flow rate.

The proposed formulas do not take into consideration the spacecraft shadowing by the planet, which interrupts the thrust if it uses solar power. However, the shadowing time can be calculated using the formulas and the interruption can be taken into account afterward such as has been done in Ref. 29.

Conclusions

The proposed formulas allow calculation of the spacecraft three-dimensional low-thrust trajectory near an oblate planet both for ascent and descent. The second approximation provides a level of accuracy that is quite satisfactory for the early phase of the mission analysis if the nondimensional thrust acceleration is much smaller than one. However, even for the case where the nondimensional thrust acceleration is close to one, the number of orbits and the longitude of the ascending node are calculated accurately; the escape time (or time of insertion into a circular orbit from outside the planet's sphere of influence) also can be calculated quite accurately (with an error within 1.3% for a wide range of the thrust and exhaust velocity values), giving an accurate estimate of the propellant consumption. Even the first approximation can be used for rough preliminary assessments; this is especially true for the first approximation, which gives quite accurate values of the longitude of the ascending node.

Appendix A: Calculation of Integral $\int_0^t (\mathrm{d}t/r^{k/2})$

Let us designate

$$x = (m/m_0) = 1 - (\beta t/m_0) \tag{A1}$$

From Eqs. (7) and (A1) we have

$$\ln x = \eta - \eta_0 \tag{A2}$$

Let us consider the integrals

$$L_k = m_0 \int_0^t (\eta_0 + \ln x)^k \, \mathrm{d}x \qquad (k = 0, 1, 2, \ldots) \tag{A3}$$

Let us first calculate the indefinite integrals

$$l_k = m_0 \int (\eta_0 + \ln x)^k \, \mathrm{d}x$$

This integral can be easily calculated by parts,

$$l_k = m_0 x(\eta_0 + \ln x)^k - m_0 \int x \, \mathrm{d}\,(\eta_0 + \ln x)^k$$

$$= m(\eta_0 + \eta - \eta_0) - k m_0 \int (\eta_0 + \ln x)^{k-1} \, \mathrm{d}x$$

Thus,

$$l_0 = m_0 x = m \tag{A4}$$

$$l_k = m\eta^k - kl_{k-1} \tag{A5}$$

From Eq. (A5) obtain l_k in another form,

$$l_k = m[\eta^k - k\eta^{k-1} + k(k-1)\eta^{k-2} - \cdots - (-1)^k k!\eta + (-1)^k k!] \tag{A6}$$

[the arbitrary constants are ignored in Eqs. (A4–A6)]. Then

$$L_0 = l_0(t) - l_0(t_0) = m - m_0 \tag{A7}$$

$$L_k = l_k(t) - l_k(t_0) = m\eta^k - m_0\eta_0^k - kL_{k-1} \tag{A8}$$

Using Eqs. (6) and (A1), we have

$$\int_0^t \frac{\mathrm{d}t}{r^{k/2}} = \frac{m_0}{r_0^{k/2}\eta_0^k \dot{m}} \int_{x_0}^x (\eta_0 + \ell_n x)^k \,\mathrm{d}x = \mp \frac{c^k L_k}{r_0^{k/2}\sqrt{(\mu/r_0)^k}\beta}$$

where − and + correspond to ascent and descent, respectively. When Eqs. (A1) and (A2) are taken into account, it can be shown that for odd k,

$$L_k < 0 \quad \text{if} \quad 0 < \eta < \eta_0 \text{ (ascent)}$$
$$L_k > 0 \quad \text{if} \quad \eta < \eta_0 < 0 \text{ (descent)}$$

and for even k,

$$L_k < 0$$

Thus, using Eq. (5), we finally obtain

$$\int_0^t \frac{\mathrm{d}t}{r^{k/2}} = \frac{c^k |L_k|}{\mu^{k/2}\beta} \tag{A9}$$

Appendix B: First-Order Value of $\dot{\Omega}$ for a Circular Orbit

Let us consider free circular motion, that is, without any thrust, near the oblate planet. As can be derived from the results of Brouwer,[28] the first-order value of $\dot{\Omega}$ for the circular orbit is

$$\dot{\Omega} = -\tfrac{3}{2}J_2\sqrt{\mu/\bar{a}^3}(R_e/\bar{a})^2\big[1 + (J_2/4)(R_e/\bar{a})^2 \times(15 - 19\sin^2\bar{i})\big]\cos\bar{i} \tag{B1}$$

Using Eq. (27), we can express the osculating semimajor axis in terms of the average orbit radius:

$$\bar{a} = \frac{1}{2/\bar{r} - \bar{v}^2/\mu} = \bar{r}\left(1 + \frac{3}{2}\frac{B}{\bar{r}^2}\right) \tag{B2}$$

Substituting Eq. (B2) into Eq. (B1), we have

$$\dot{\Omega} = -\frac{3}{2}J_2\sqrt{\frac{\mu}{\bar{r}^3}}\left(\frac{R_e}{\bar{r}}\right)^2\left[1 + \frac{J_2}{4}\left(\frac{R_e}{\bar{r}}\right)^2(15 - 19\sin^2\bar{i})\right] \times\left(1 - \frac{21}{4}\frac{B}{\bar{r}^2}\right)\cos\bar{i} = -\frac{3}{2}J_2\sqrt{\frac{\mu}{\bar{r}^3}}\left(\frac{R_e}{\bar{r}}\right)^2\left[1 - \frac{3}{2}J_2\left(\frac{R_e}{\bar{r}}\right)^2 \times\left(1 - \frac{25}{12}\sin^2\bar{i}\right)\right]\cos\bar{i} \tag{B3}$$

It is also convenient to obtain $\dot{\Omega}$ via the parameters of the circular motion at the ascending node. The circular velocity v_Ω can be found for given values r_Ω and i_Ω from Eqs. (22), (23), and (28) where $u = 0$:

$$v_\Omega = \sqrt{\mu/r_\Omega}\big[1 + \tfrac{3}{4}J_2(R_e/r_\Omega)^2\big(1 - \tfrac{1}{2}\sin^2 i_\Omega\big)\big] \tag{B4}$$

It can be shown that the values r_Ω and v_Ω are equal to the average values $\bar{r}$ and $\bar{v}$, respectively, that is, to the values averaged over one orbit. However, i_Ω is not equal to the average value $\bar{i}$ that is used in Eq. (B1). It can be concluded from the short-period perturbation of the inclination[30] that

$$\bar{i} = i_\Omega - \tfrac{3}{4}J_2(R_e/r_\Omega)^2\sin i_\Omega\cos i_\Omega$$

$$\cos\bar{i} = \cos i_\Omega\big[1 + \tfrac{3}{4}J_2(R_e/r_\Omega)^2\sin^2 i_\Omega\big] \tag{B5}$$

Note that the short-period perturbation of the inclination was first obtained by Brouwer, although the expression in Ref. 28 contained the incorrect factor $\frac{1}{2}$ instead of $\frac{1}{4}$. Substituting Eq. (B5) into Eq. (B3) and neglecting terms of $\mathcal{O}(J_2^3)$, we have

$$\dot{\Omega} = -\frac{3}{2}J_2\sqrt{\frac{\mu}{r_\Omega^3}}\left(\frac{R_e}{r_\Omega}\right)^2\left[1 - \frac{3}{2}J_2\left(\frac{R_e}{r_\Omega}\right)^2\left(1 - \frac{25}{12}\sin^2 i_\Omega\right)\right] \times\left[1 + \frac{3}{4}J_2\left(\frac{R_e}{r_\Omega}\right)^2\sin^2 i_\Omega\right]\cos i_\Omega = -\frac{3}{2}J_2\sqrt{\frac{\mu}{r_\Omega^3}}\left(\frac{R_e}{r_\Omega}\right)^2 \times\left[1 - \frac{3}{2}J_2\left(\frac{R_e}{r_\Omega}\right)^2\left(1 - \frac{31}{12}\sin^2 i_\Omega\right)\right]\cos i_\Omega \tag{B6}$$

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References

[1]Tsien, H. S., "Takeoff from Satellite Orbit," *Jet Propulsion*, Vol. 23, 1953, pp. 233–236.

[2]Benney, D. J., "Escape from a Circular Orbit Using Tangential Thrust," *Jet Propulsion*, Vol. 28, No. 3, 1958, pp. 167–169.

[3]Melbourne, W. G., "Interplanetary Trajectories and Payload Capabilities of Advanced Propulsion Vehicles," Jet Propulsion Lab., JPL TR 32-68, California Inst. of Technology, Pasadena, CA, March 1961.

[4]Faulders, C. R., "Optimum Thrust Programming of Electrically Powered Rocket Vehicles for Earth Escape," *AIAA Journal*, Vol. 1, No. 1, 1963, pp. 233–234.

[5]Hinz, H. K., "Optimal Low-Thrust Near-Circular Orbital Transfer," *AIAA Journal*, Vol. 1, No. 6, 1963, pp. 1367–1371.

[6]Beletsky, V. V., and Egorov, V. A., "The Acceleration of a Spacecraft Within the Range of Planetary Influence," *Cosmic Research*, Vol. 2, No. 3, 1964, pp. 331–345.

[7]Okhotsimsky, D. E., "Investigation of Motion in Central Field Under Influence of Constant Tangential Acceleration," *Cosmic Research*, Vol. 2, No. 6, 1964, pp. 715–736.

[8]Ting, L., and Brofman, S., "Take-Off from Circular Orbit by Small Thrust," *Zeitschrift für Angerwandte Mathematik und Mechanik*, Vol. 44, 1964, pp. 417–428.

[9]Nayfeh, A. H., "Take-Off from a Circular Orbit by a Small Thrust," AIAA Paper 65-0688, 1965.

[10]Cohen, M. J., "Low-Thrust Spiral Trajectory of a Satellite of Variable Mass," *AIAA Journal*, Vol. 3, No. 3, 1965, pp. 1946–1949.

[11]Edelbaum, T. N., "Optimum Power-Limited Orbit Transfer in Strong Gravity Field," *AIAA Journal*, Vol. 3, No. 5, 1965, pp. 921–925.

[12]Evtushenko, I. G., "Influence of Tangential Acceleration on the Motion of a Satellite," *Journal of Applied Mathematics and Mechanics*, Vol. 30, No. 3, 1967, pp. 710–716; also *Prikladnaya Matematika y Mekanika*, Vol. 30, No. 3, 1966, pp. 594–598 (in Russian).

[13]Yun, Y. S., and Eckstein, M. C., "Ascent or Descent from Satellite Orbit by Low Thrust," *AIAA Journal*, Vol. 4, No. 12, 1966, pp. 2203–2209.

[14]Brofman, W., "Approximate Analytical Solution for Satellite Orbits Subjected to Small Thrust or Drag," *AIAA Journal*, Vol. 5, No. 6, 1967, pp. 1121–1128.

[15]Eckstein, M. C., and Yun, Y. S., "Low-Thrust Elliptic Spiral Trajectories of a Satellite of Variable Mass," *AIAA Journal*, Vol. 5, No. 8, 1967, pp. 1491–1494.

[16]Zee, C.-H., "Comments on 'Approximate Analytical Solution for Satellite Orbits Subjected to Small Thrust or Drag,'" *AIAA Journal*, Vol. 5, No. 11, 1967, pp. 2111–2112.

[17]Zee, C.-H., "Improved Tangential Thrust Trajectories—Improved First-Order Solution," *AIAA Journal*, Vol. 6, No. 7, 1968, pp. 1378–1379.

[18]Lebedev, V. N., "Calculation of Motion of Spacecraft with Low Thrust," *Mathematical Methods in Spacecraft Dynamics*, No. 5, Computing Center of the USSR Academy of Sciences, Moscow, 1968 (in Russian).

[19]Efimov, G. B., "Optimal Acceleration to Hyperbolic Velocities," *Cosmic Research*, Vol. 8, No. 1, 1970, pp. 23–44.

[20]Jacobson, R. A., and Powers, W. F., "Asymptotic Solution to the Problem of Optimal Low-Thrust Energy Increase," *AIAA Journal*, Vol. 10, No. 12, 1972, pp. 1679–1680.

[21]Marec, J.-P., and Vihn, N. X., "Optimal Low-Thrust, Limited Power Transfers Between Arbitrary Elliptical Orbits," *Acta Astronautica*, Vol. 4, 1977, pp. 511–540.

[22]Da Silva Fernandes, S., and Sessin, W., "Optimal Low-Thrust Limited Power Transfer Between Neighbouring Quasi-Circular Orbits of Small Inclination Around an Oblate Planet," *Acta Astronautica*, Vol. 19, No. 5, 1989, pp. 401–409.

[23]Da Silva Fernandes, S., "Optimal Low-Thrust Transfer Between Neighbouring Quasi-Circular Orbits Around an Oblate Planet," *Acta Astronautica*, Vol. 19, No. 12, 1989, pp. 933–938.

[24]Alfano, S., and Thorne, J. D., "Circle-to-Circle Constant Thrust Orbit Raising," *Journal of Astronautical Sciences*, Vol. 42, No. 1, 1994, pp. 35–45.

[25]Da Silva Fernandes, S., "Optimum Low-Thrust Limited Power Transfers Between Neighbouring Elliptic Non-Equatorial Orbits in a Non-Central Gravity Field," *Acta Astronautica*, Vol. 35, No. 12, 1995, pp. 763–770.

[26]Geffroy, S., and Epenoy, R., "Optimal Thrust Transfers with Constraints—Generalization of Averaging Technique," *Acta Astronautica*, Vol. 41, No. 3, 1997, pp. 133–149.

[27]Thorne, J. D., and Hall, C. D., "Minimum-Time Continuous-Thrust Orbit Transfers," *Journal of Astronautical Sciences*, Vol. 45, No. 4, 1997, pp. 411–432.

[28]Brouwer, D., "Solution of the Problem of Artificial Satellite Theory Without Atmospheric Drag," *Astronomical Journal*, Vol. 64, No. 11, 1959, pp. 378–396.

[29]Sukhanov, A. A., Eismont, N. A., Prudkoglyad, A. V., "Trajectory Design for Experimental Mission to Sun–Earth L_1 and L_2 Points Using SEP," *Journal of the Brazilian Society Mechanical Sciences*, Vol. 21, Special Issue, 1999, pp. 313–321.

[30]Kozai, Y., "Second-Order Solution of Artificial Satellite Theory Without Air Drag," *Astronomical Journal*, Vol. 67, No. 7, 1962, pp. 446–461.